Appendix B

The Scalar or Dot Product

The multiplication of a vector by a scalar was discussed in Appendix A. When we multiply a vector by another vector, we must define precisely what we mean. One type of vector product is called the *scalar* or *dot* product and is covered in this appendix. A second type of vector product is called the *vector* or *cross* product and is covered in Appendix C.

Prerequisite knowledge:
Appendix A – Addition and Subtraction of Vectors

B.1 Definition of the Dot Product

The *scalar* or *dot* product and is written as $\mathbf{A} \bullet \mathbf{B}$ and read "**A** dot **B**". The dot product is defined by the relation

$$\mathbf{A} \bullet \mathbf{B} = AB\cos\phi \tag{B.1}$$

where ϕ is the angle between **A** and **B**. Since the dot product $AB\cos\phi$ has only a magnitude and not a direction, then $\mathbf{A} \bullet \mathbf{B}$ is a *scalar* quantity.

The dot product $\mathbf{A} \bullet \mathbf{B} = AB\cos\phi$ can be written as $\mathbf{A} \bullet \mathbf{B} = \left(A\cos\phi\right)B$ where $A\cos\phi$ is the magnitude of the projection of **A** on **B** as shown in Fig. B.1.

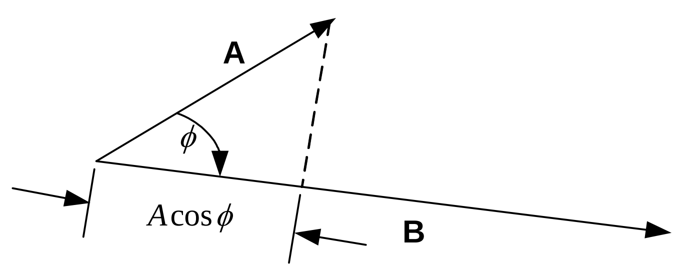

Figure B.1 $\mathbf{A} \bullet \mathbf{B} = \left(A\cos\phi\right)B$

The dot product can also be written as $\mathbf{A} \bullet \mathbf{B} = A\left(B\cos\phi\right)$ where $B\cos\phi$ is the magnitude of the projection of **B** on **A** as shown in Fig. B.2.

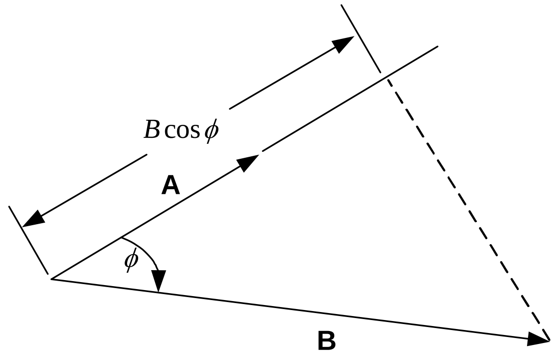

Figure B.2 $\mathbf{A} \bullet \mathbf{B} = A\left(B\cos\phi\right)$

From the definition of the dot product, $\mathbf{A} \bullet \mathbf{B} = AB\cos\phi$ and $\mathbf{B} \bullet \mathbf{A} = BA\cos\phi$. It is therefore clear that $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$ and the commutative law holds for the scalar product. The distributive law $\mathbf{A} \bullet \left(\mathbf{B} + \mathbf{C}\right) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$ also holds and is illustrated for the special case shown in Fig. B.3 where **D** = **B** + **C**.

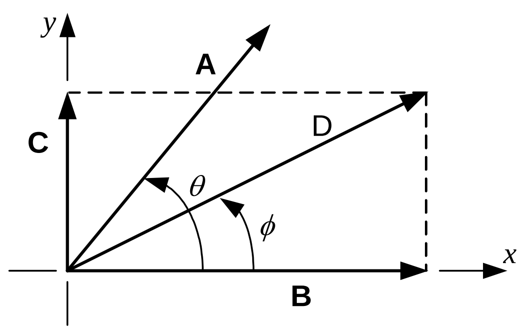

Figure B.3 $\mathbf{A} \bullet \left(\mathbf{B} + \mathbf{C}\right) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$

Then

$$\begin{aligned}
\mathbf{A} \bullet \left(\mathbf{B} + \mathbf{C}\right) &= \mathbf{A} \bullet \mathbf{D} \\
&= AD\cos(\theta - \phi) \\
&= AD\left(\cos\theta\cos\phi + \sin\theta\sin\phi\right)
\end{aligned}$$

But since $B = D\cos\phi$ and $C = D\sin\phi$

$$\begin{aligned}
\mathbf{A} \bullet \left(\mathbf{B} + \mathbf{C}\right) &= AB\cos\theta + AC\sin\theta \\
&= \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}
\end{aligned}$$

since the angle between **A** and **C** is $\frac{\pi}{2} - \theta$ and $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$.

B.2 Dot Product and Vector Components

The form of the dot product can be written conveniently in terms of its components in a rectangular coordinate system. Consider the two-dimensional case shown in Fig. B.4.

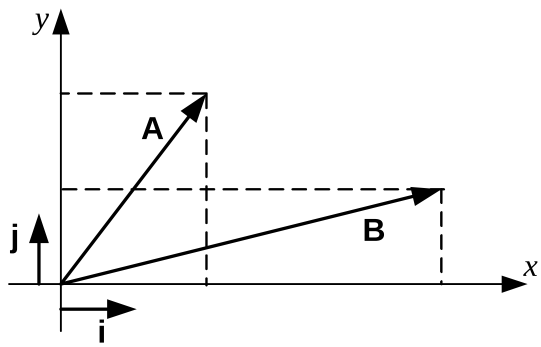

Figure B.4 Dot product in a rectangular coordinate system

The vectors **A** and **B** can be written in the component form $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j}$. Then

$$\begin{aligned}\mathbf{A} \bullet \mathbf{B} &= \left(A_x\mathbf{i} + A_y\mathbf{j}\right) \bullet \left(B_x\mathbf{i} + B_y\mathbf{j}\right) \\ &= A_xB_x\mathbf{i} \bullet \mathbf{i} + A_yB_x\mathbf{j} \bullet \mathbf{i} + A_xB_y\mathbf{i} \bullet \mathbf{j} + A_yB_y\mathbf{j} \bullet \mathbf{j}\end{aligned}$$

Since the unit vectors **i** and **j** are orthogonal (i.e., perpendicular), then from the definition of the scalar product $\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{i} = 0$. Thus, the scalar product can be written as

$$\mathbf{A} \bullet \mathbf{B} = A_xB_x + A_yB_y \qquad \text{(B.2)}$$

Note that the dot product $\mathbf{A} \bullet \mathbf{B}$ must always involve the product of two *vectors*, and since the result is a *scalar*, then an expression such as $\mathbf{A} \bullet \left(\mathbf{B} \bullet \mathbf{C}\right)$ has no meaning. On the other hand the expression $\mathbf{A}\left(\mathbf{B} \bullet \mathbf{C}\right)$ does have a meaning.

B.3 Dot Product Properties

Consider the vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$. From the orthogonality of the unit vectors described in Section B.2 it follows that

$$\mathbf{A} \bullet \mathbf{i} = A_x$$
$$\mathbf{A} \bullet \mathbf{j} = A_y$$
$$\mathbf{A} \bullet \mathbf{k} = A_z$$

Consider the definition of the dot product $\mathbf{A} \bullet \mathbf{B} = AB\cos\phi$. If **B** = **A**, then $\mathbf{A} \bullet \mathbf{A} = A^2$.

B.4 Example B1

Given the vectors $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ find $\mathbf{A} \bullet \mathbf{B}$.

Answer 1:

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$
$$= (1)(3) + (-2)(1) + (4)(-2)$$
$$= 3 - 2 - 8 = -7$$

Answer 2: In Matlab the dot product of vectors **A** and **B** can be written as *dot*(**A,B**) as shown in Matlab Example B1.

Matlab Example B1

```
>> A = [1 -2 4]

A =
     1    -2     4

>> B = [3 1 -2]

B =
     3     1    -2

>> AdotB = dot(A,B)

AdotB =
    -7

>>
```

B.5 Example B2

Find the angle between the vectors **A** and **B** in Example B1.

Answer 1:

$$\mathbf{A} \bullet \mathbf{B} = AB\cos\phi = -7$$
$$A = \sqrt{21} \qquad B = \sqrt{14}$$
$$\cos\phi = \frac{\mathbf{A} \bullet \mathbf{B}}{AB} = \frac{-7}{\sqrt{21}\sqrt{14}} = -0.408$$
$$\phi = 114^\circ$$

Answer 2: In Matlab the solution can be found by writing the single Matlab equation shown in Matlab Example B2.

Matlab Example B2

```
>> A = [1 -2 4]

A =
     1    -2     4

>> B = [3 1 -2]

B =
     3     1    -2

>> phi = (acos(dot(A,B)/(norm(A)*norm(B))))*180/pi

phi =

  114.0948

>>
```

Note carefully the need to use parentheses in the equation for *phi*. The Matlab function *acos* for the arc cosine gives the answer in radians. Thus, that result must be multiplied by 180/π to give the answer in degrees.

Problems

Where appropriate use Matlab to find the answers to the following problems.

B-1 Given the vectors $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ find

(a) Find $\mathbf{A} \bullet \mathbf{B}$ and $\mathbf{B} \bullet \mathbf{A}$.

(b) Find the smaller angle between **A** and **B**.

(c) What is the component of **A** in the direction of **B**? (d) What is the component of **B** in the direction of **A**?

B-2 If $\mathbf{A} = 10\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, determine A^2.

B-3 Given the vectors $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$ show that **A** and **B** are perpendicular to each other.

B-4 $\mathbf{A} \bullet \mathbf{i} = 3$, $\mathbf{A} \bullet \mathbf{j} = 5$, and $\mathbf{A} \bullet \mathbf{k} = -2$. Find **A**.

z

B-5 If $\mathbf{A} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ find, $(2\mathbf{A} + \mathbf{B}) \bullet (\mathbf{A} - 2\mathbf{B})$.

B-6 For what values of α are vectors $\mathbf{A} = \alpha\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 2\alpha\mathbf{i} + \alpha\mathbf{j} - 4\mathbf{k}$ perpendicular?